DESIGN AND ANALYSIS OF ALGORITHMS

Merge Sort

$O(n^2)$ sorting algorithms

- Selection sort and insertion sort are both $O(n^2)$
- $O(n^2)$ sorting is infeasible for n over 100000

A different strategy?

- Divide array in two equal parts
- Separately sort left and right half
- Combine the two sorted halves to get the full array sorted

Combining sorted lists

- Given two sorted lists A and B, combine into a sorted list C
  - Compare first element of A and B
  - Move it into C
  - Repeat until all elements in A and B are over
- Merging A and B

Merging two sorted lists

32	74	89
21	55	64

Merging two sorted lists

32	74	89
~~21~~	55	64
21		

Merging two sorted lists

~~32~~	74	89
~~21~~	55	64
21	32	

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~~32~~	74	89
~~21~~	~~55~~	64
21	32	55

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~~32~~	74	89	
~~21~~	~~55~~	~~64~~	
21	32	55	64

Merging two sorted lists

~~32~~ ~~74~~ 89

~~21~~ ~~55~~ ~~64~~

21 32 55 64 74

Merging two sorted lists

~~32~~ ~~74~~ ~~89~~

~~21~~ ~~55~~ ~~64~~

21 32 55 64 74 89

Merge Sort

- Sort A[0] to A[n/2-1]
- Sort A[n/2] to A[n-1]
- Merge sorted halves into B[0..n-1]
- How do we sort the halves?
    - Recursively, using the same strategy!

Merge Sort

43	32	22	78	63	57	91	13

Merge Sort

43	32	22	78	63	57	91	13

43	32	22	78

Merge Sort

43	32	22	78	63	57	91	13

43	32	22	78

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Merge Sort

43	32	22	78	63	57	91	13

43	32	22	78

63	57	91	13

43	32

22	78

Merge Sort

43	32	22	78	63	57	91	13

43	32	22	78

63	57	91	13

43	32

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91	13

Merge Sort

43	32	22	78	63	57	91	13

43	32	22	78

63	57	91	13

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32

Merge Sort

43	32	22	78	63	57	91	13

43	32	22	78

63	57	91	13

43	32

22	78

63	57

91	13

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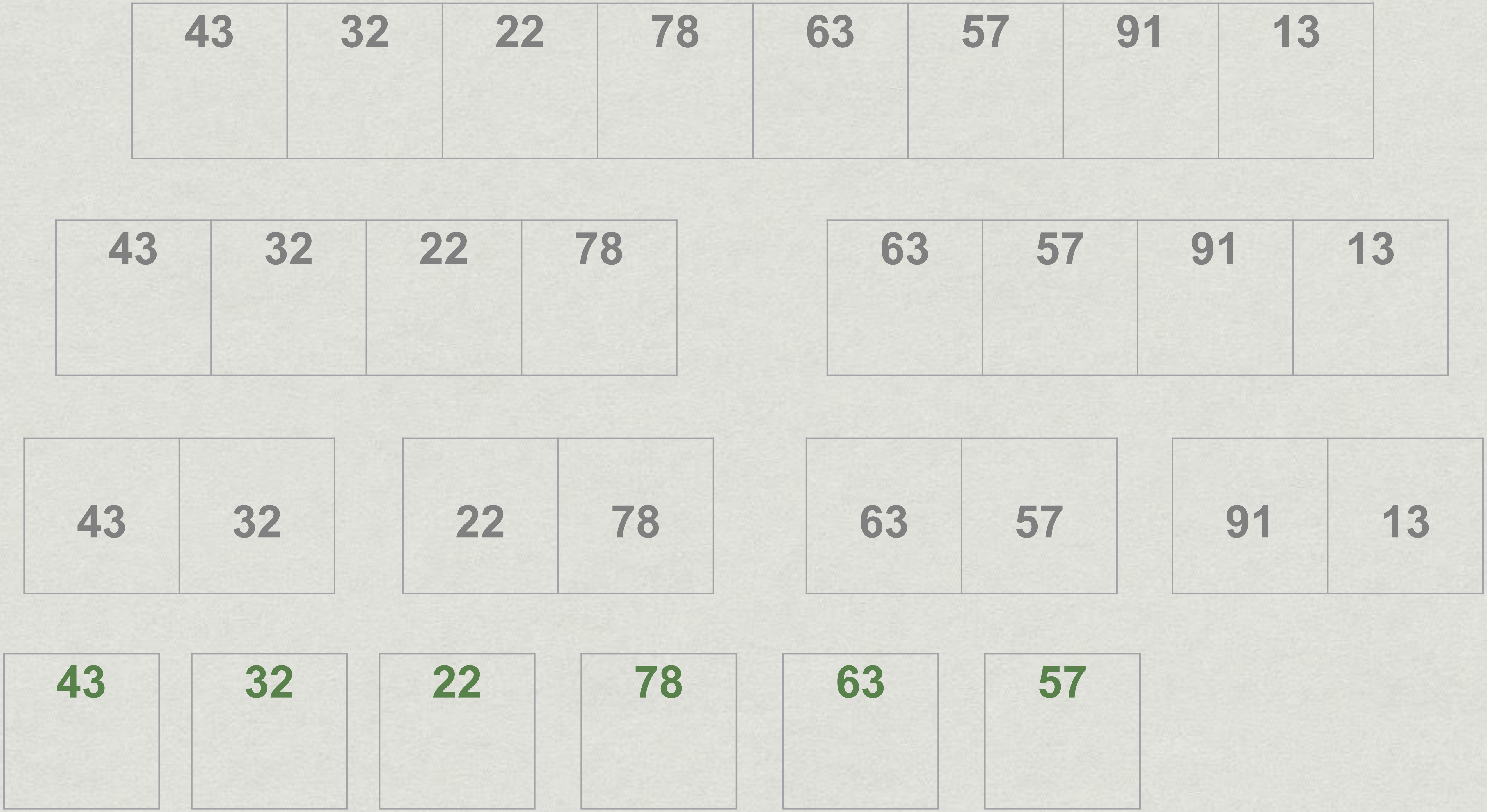
78

Merge Sort



43 32 22 78 63 57 91 13
43 32 22 78
63 57 91 13
43 32
22 78
63 57
91 13
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57

Merge Sort

43	32	22	78	63	57	91	13

43	32	22	78

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Merge Sort

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Merge Sort

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Merge Sort

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22	78

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Divide and conquer

- Break up problem into disjoint parts
- Solve each part separately
- Combine the solutions efficiently

Merging sorted lists

Combine two sorted lists A and B into C

- If A is empty, copy B into C
- If B is empty, copy A into C
- Otherwise, compare first element of A and B and move the smaller of the two into C
- Repeat until all elements in A and B have been moved

Merging

```
function Merge(A,m,B,n,C)
  // Merge A[0..m-1], B[0..n-1] into C[0..m+n-1]

  i = 0; j = 0; k = 0;
  // Current positions in A,B,C respectively

  while (k < m+n)
  // Case 0: One of the two lists is empty
     if (i==m) {j++; k++;}
     if (j==n) {i++; k++;}
  // Case 1: Move head of A into C
     if (A[i] <= B[j]) {C[k] = B[j]; j++; k++;}
  // Case 2: Move head of B into C
     if (A[i] > B[j]) {C[k] = B[j]; j++; k++;}
```

Merge Sort

To sort A[0..n-1] into B[0..n-1]

- If n is 1, nothing to be done
- Otherwise
  - Sort A[0..n/2-1] into L (left)
  - Sort A[n/2..n-1] into R (right)
  - Merge L and R into B

Merge Sort

```
function MergeSort(A,left,right,B)
    // Sort the segment A[left..right-1] into B

    if (right - left == 1) // Base case
        B[0] = A[left]

    if (right - left > 1)  // Recursive call

        mid  =  (left+right)/2

        MergeSort(A,left,mid,L)
        MergeSort(A,mid,right,R)

        Merge(L,mid-left,R,right-mid,B)
```

DESIGN AND ANALYSIS OF ALGORITHMS

Merge sort: Analysis

Merging sorted lists

Combine two sorted lists A and B into C

- If A is empty, copy B into C
- If B is empty, copy A into C
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Merging

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function Merge(A,m,B,n,C)
    // Merge A[0..m-1], B[0..n-1] into C[0..m+n-1]

    i = 0; j = 0; k = 0;
    // Current positions in A,B,C respectively

    while (k < m+n)
        // Case 1: Move head of A into C
        if (j==n or A[i] <= B[j])
            C[k] = A[i]; i++; k++

        // Case 2: Move head of B into C
        if (i==m or A[i] > B[j])
            C[k] = B[j]; j++; k++
```

Analysis of Merge

How much time does Merge take?

- Merge A of size m, B of size n into C
- In each iteration, we add one element to C
  - At most 7 basic operations per iteration
  - Size of C is m+n
  - $m+n \lesssim 2 \max(m,n)$
- Hence $O(\max(m,n)) = O(n)$ if $m \approx n$

Merge Sort

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- If n is 1, nothing to be done
- Otherwise
  - Sort A[0..n/2-1] into L (left)
  - Sort A[n/2..n-1] into R (right)
  - Merge L and R into B

Analysis of Merge Sort …

- $t(n)$: time taken by Merge Sort on input of size n
  - Assume, for simplicity, that $n = 2^k$
- $t(n) = 2t(n/2) + n$
  - Two subproblems of size $n/2$
  - Merging solutions requires time $O(n/2+n/2) = O(n)$
- Solve the recurrence by unwinding

Analysis of Merge Sort …

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  $= 2 [\ 2t(n/4) + n/2\ ] + n = 2^2\, t(n/2^2) + 2n$

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  $= 2^2 [2t(n/2^3) + n/2^2] + 2n = 2^3 t(n/2^3) + 3n$

  …

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  …

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Analysis of Merge Sort …

-

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  When $j = \log n$, $n/2^j = 1$, so $t(n/2^j) = 1$

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- When $j = \log n$, $n/2^j = 1$, so $t(n/2^j) = 1$
  - $\log n$ means $\log_2 n$ unless otherwise specified!

Analysis of Merge Sort …

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- When $j = \log n$, $n/2^j = 1$, so $t(n/2^j) = 1$
  - log n means $\log_2 n$ unless otherwise specified!
- $t(n) = 2^j t(n/2^j) + jn = 2^{\log n} + (\log n) n = n + n \log n = O(n \log n)$

O(n log n) sorting

- Recall that O(n log n) is much more efficient than $O(n^2)$
- Assuming 10^8 operations per second, feasible input size goes from 10,000 to 10,000,000 (10 million or 1 crore)

Variations on merge

Union of two sorted lists (discard duplicates)

- If A[i] == B[j], copy A[i] to C[k] and increment i,j,k
  - Intersection of two sorted lists
- If A[i] < B[j], increment i
  - If B[j] < A[i], increment j
  - If A[i] == B[j], copy A[i] to C[k] and increment i,j,k
  - Exercise:

  List difference: elements in A but not in B

Merge Sort: Shortcomings

- Merging A and B creates a new array C
  - No obvious way to efficiently merge in place
- Extra storage can be costly
- Inherently recursive
  - Recursive call and return are expensive

Alternative approach

- Extra space is required to merge
- Merging happens because elements in left half must move right and vice versa
- Can we divide so that everything to the left is smaller than everything to the right?
  - No need to merge!